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Small resolutions and non-liftable Calabi-Yau threefolds

Abstract. We use properties of small resolutions of the ordinary double point in dimension three to construct smooth non-liftable Calabi-Yau threefolds. In particular, we construct a smooth projective Calabi-Yau threefold over $\mathbb{F}_3$ that does not lift to characteristic zero and a smooth projective Calabi-Yau threefold over $\mathbb{F}_5$ having an obstructed deformation. We also construct many examples of smooth Calabi-Yau algebraic spaces over $\mathbb{F}_p$ that do not lift to algebraic spaces in characteristic zero.

1. Introduction

Hirokado [12] and Schröer [21] have constructed examples of projective Calabi-Yau threefolds in characteristic 2 and 3 that have no liftings to characteristic zero. The question arises if there exist non-liftable examples in higher characteristic [7,9]. In this paper we construct further examples of non-liftable Calabi-Yau threefolds, most notably a rigid Calabi-Yau threefold in characteristic 3 and a projective Calabi-Yau threefold in characteristic 5 with an obstructed deformation. Using our method, it is easy to produce non-liftable Calabi-Yau threefolds in higher characteristic in the category of algebraic spaces. Up to now, we were unable to find further projective examples.

  Our method exploits a remarkable feature of birational geometry in dimension ≥ 3, namely the appearance of singularities that have a *small resolution*, that is, admit a map $\pi : Y \longrightarrow X$ such that the exceptional set has codimension ≥ 2 in Y. The key example of the ordinary double point in dimension three was described by Atiyah [2]. It admits two different small resolutions that contract a single rational curve with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Such a rational curve is a stable submanifold in the sense that it lifts to any deformation of the ambient variety

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[18]. As a consequence, blowing down deformations of such small resolutions give deformations of the variety which remain singular [8]. This is very different form the geometry related to the ordinary double point in dimension two. We use this peculiar property of the three dimensional ordinary double point to give a criterion for varieties in characteristic p to have no lifting to characteristic zero. The criterion can be applied to certain rigid Calabi-Yau threefolds that can be obtained as resolution of a double cover of $\mathbb{P}^3$, ramified over a reducible hypersurface of degree eight and fibre products of rational elliptic surfaces.

2. Generalities on liftings

Consider a scheme $f : X \longrightarrow T$ flat over T and assume $T \hookrightarrow S$ embeds T as a closed subscheme in S. By a *lifting* of X to S we mean a scheme $F : \mathcal{X} \longrightarrow S$ flat over S, such that $X = \mathcal{X} \times_S T$. We are mainly interested in the case $T = Spec(k)$, where k is a field of characteristic $p > 0$ and S is artinian local ring with k as residue field. Such liftings are usually constructed by a stepwise process from situations where $T \subset S$ is defined by an ideal I of square zero, which will be assumed from now on. One could in fact reduce to the case where $T \hookrightarrow S$ is a *simple extension*, that is, if S is defined by an ideal I that is annihilated by the maximal ideal of S.

The liftings of $f : X \longrightarrow T$ to S form a category (in fact a groupoid) where morphisms between two liftings are defined to be an isomorphism over S, which restricts to the identity over T.

If $f : X \longrightarrow T$ is *smooth*, then liftings always exist locally; the local lifts of $U \subset X$ form a torsor over $\Theta_{U/T} := Hom_U(\Omega_{U/T}, \mathcal{O}_U)$. There is an obstruction element $Ob(X/T, S) \in H^2(X, \Theta_{X/T} \otimes f^*I)$ that vanishes precisely when $f : X \longrightarrow T$ lifts to S. If $Ob(X/T, S) = 0$, then the set of isomorphism classes of liftings of $X \longrightarrow T$ is a torsor over $H^1(X, \Theta_{X/T} \otimes f^*I)$. The standard reference for this material is [10] or Exposé III in SGA1 [11].

In the more general case where $f : X \longrightarrow T$ is not necessarily smooth, one has to use the theory of the cotangent complex $\mathbb{L}_{X/T}^{\cdot}$, the derived version of $\Omega_{X/T}$ [16]. For an $\mathcal{O}_X$-module $\mathcal{F}$ we put $\mathbb{T}^i(X/T, \mathcal{F}) := Ext_X^i(\mathbb{L}_{X/T}^{\cdot}, \mathcal{F}) = \mathbb{H}^i(Hom_X(\mathbb{L}_{X/T}^{\cdot}, \mathcal{F}))$. Then one has the following fundamental fact:

Theorem 2.1. [17, Theorem 5.31] *Let $T \hookrightarrow S$ be defined by an ideal I of square zero and $f : X \longrightarrow T$ a flat scheme over T. Then there is an obstruction*

$$Ob(X/T, S) \in \mathbb{T}^2(X/T, f^*I)$$

It is zero precisely if $f : X \longrightarrow T$ can be lifted to S. If $Ob(X/T, S) = 0$, then the set of liftings of $f : X \longrightarrow T$ is a torsor over $\mathbb{T}^1(X/T, f^*I)$. The group of automorphisms of any lifting is isomorphic to $\mathbb{T}^0(X/T, f^*I)$.

We refer to [16], III 2 for details. Note that in the case of a simple extension the relevant groups are isomorphic to $\mathbb{T}^i(X/T)$ ($i = 0, 1, 2$), where we put $\mathbb{T}^i(X/T) := \mathbb{T}^i(X/T, \mathcal{O}_X)$.

3. Blowing down liftings

The phenomenon of simultaneous resolution was first observed for the simple singularities by Brieskorn [4]. An important feature of the situation is that under mild cohomological conditions deformations or liftings can be "blown-down". We formulate and sketch a proof of a variant of a theorem formulated in [24].

Theorem 3.1. *Let $\pi : Y \longrightarrow X$ be a morphism of schemes over k and let $S = Spec(A)$, A artinian with residue field k. Assume that $\mathcal{O}_X = \pi_*(\mathcal{O}_Y)$ and $R^1\pi_*(\mathcal{O}_Y) = 0$. Then for every lifting $\mathcal{Y} \longrightarrow S$ of Y there exists a preferred lifting $\mathcal{X} \longrightarrow S$ making a commutative diagram*

$$
\begin{array}{ccc}
Y & \hookrightarrow & \mathcal{Y} \\
\downarrow & & \downarrow \\
X & \hookrightarrow & \mathcal{X}
\end{array}
$$

Proof. A lifting of X over S is defined by giving for each affine open subset $U \subset X$ a flat $\mathcal{O}_S$-algebra, reducing to $\mathcal{O}_U$ over k. These algebras should be compatible in the sense that isomorphisms are given for the restrictions to overlaps, which satisfy a cocycle condition. Let $V := \pi^{-1}(U) \subset Y$. If a lifting $\mathcal{Y} \longrightarrow S$ is given, we obtain an $\mathcal{O}_S$-algebra $A(U) := H^0(V, \mathcal{O}_\mathcal{Y})$.

The vanishing of $R^1\pi_*(\mathcal{O}_Y)$ implies that $H^1(V, \mathcal{O}_Y) = 0$. By reducing to the case of a simple extension and using the long exact cohomology sequence, one sees that this implies that $A(U)$ is $\mathcal{O}_S$-flat, whereas $\pi_*(\mathcal{O}_Y) = \mathcal{O}_X$ shows that $Spec(A(U))$ indeed is a lift of U over S. The compatibilities for gluing are implied by the corresponding properties for $\mathcal{Y}$.   □

Next we want to compare the groups $\mathbb{T}^i(Y/T)$ and $\mathbb{T}^i(X/T)$. Recall that these are hypercohomology groups of the cotangent complex, which can be calculated using a local-to-global spectral sequence that reads

$$
E_2^{i,j} = H^i\left(X, \mathcal{T}_{X/T}^j\right) \Longrightarrow \mathbb{T}^{i+j}(X/T)
$$

Here $\mathcal{T}_{X/T}^j$ are the cohomology sheaves of the cotangent complex, so $\mathcal{T}_{X/T}^0 = \Theta_{X/T}$. The sheaf $\mathcal{T}_{X/T}^1$ determines the isomorphism classes of local first-order deformations and $\mathcal{T}_{X/T}^2$ the local obstructions.

Proposition 3.2. *Let $\pi : Y \longrightarrow X$ be a morphism of schemes over T, Y smooth. If $\Theta_{X/T} = \pi_*(\Theta_{Y/T})$ and $R^1\pi_*(\Theta_{Y/T}) = 0$ then there is an injection*

$$
\mathbb{T}^1(Y/T) \hookrightarrow \mathbb{T}^1(X/T)
$$

Proof. From the Leray spectral sequence

$$
E_2^{ij} = H^i\left(X, R^j\pi_*(\Theta_{Y/T})\right) \Longrightarrow H^{i+j}(Y, \Theta_{Y/T})
$$

one obtains an isomorphism $H^0(Y, \Theta_{Y/T}) \simeq H^0(X, \pi_*(\Theta_{Y/T}))$ and an exact sequence

$$
0 \longrightarrow H^1\left(X, \pi_*(\Theta_{Y/T})\right) \longrightarrow H^1(Y, \Theta_{Y/T}) \longrightarrow H^0\left(X, R^1\pi_*(\Theta_{Y/T})\right) \longrightarrow \cdots
$$

As by assumption $\Theta_{X/T} = \pi_*(\Theta_{Y/T})$ and $R^1\pi_*(\Theta_{Y/T}) = 0$, we obtain an isomorphism

$$H^1(X, \Theta_{X/T}) \simeq H^1(Y, \Theta_{Y/T})$$

From the local-to-global spectral sequence of deformations one obtains

$$0 \longrightarrow H^1(X, \Theta_{X/T}) \longrightarrow \mathbb{T}^1(X) \longrightarrow H^0(X, \mathcal{T}^1_{X/T}) \longrightarrow \cdots$$

and hence $\mathbb{T}^1(Y/T) = H^1(Y, \Theta_{Y/T}) = H^1(X, \Theta_{X/T})$ injects into $\mathbb{T}^1(X/T)$.   □

Corollary 3.3. *Let* $\pi : Y \longrightarrow X$ *be a morphism of schemes over* T. *Assume that*:

(1) $\pi_*(\mathcal{O}_Y) = \mathcal{O}_Y$ *and* $R^1\pi_*(\mathcal{O}_Y) = 0$.
(2) $\Theta_{X/T} = \pi_*(\Theta_{Y/T})$ *and* $R^1\pi_*(\Theta_{Y/T}) = 0$.

Then there is at most one lifting $\mathcal{Y} \longrightarrow S$ *of Y which lies over a given lifting* $\mathcal{X} \longrightarrow S$ *of X.*

Proof. One reduces to the case of simple extensions. As the set of liftings is a torsor over $\mathbb{T}^1$, it is sufficient to show that there is an injection $\mathbb{T}^1(Y/T) \hookrightarrow \mathbb{T}^1(X/T)$, which is the content of the previous proposition.   □

Remark 3.4. If $\Theta_{X/T} = \pi_*(\Theta_{Y/T})$ and $R^i\pi_*(\Theta_{Y/T}) = 0$ for $i = 1, 2$ then one can combine the Leray spectral sequence for $\pi : Y \longrightarrow X$ and the local-to-global spectral sequences of deformations into a diagram

$$
\begin{array}{ccccccccc}
0 \longrightarrow & H^1(\Theta_{Y/T}) & \overset{\simeq}{\longrightarrow} & \mathbb{T}^1(Y/T) & \longrightarrow & 0 & \longrightarrow & H^2(\Theta_{Y/T}) & \overset{\simeq}{\longrightarrow} & \mathbb{T}^2(Y/T) \longrightarrow & 0 \\
& \simeq\downarrow & & \downarrow & & & & \simeq\downarrow & & \downarrow \\
0 \longrightarrow & H^1(\Theta_{X/T}) & \longrightarrow & \mathbb{T}^1(X/T) & \longrightarrow & H^0(\mathcal{T}^1_{X/T}) & \longrightarrow & H^2(\Theta_{X/T}) & \longrightarrow & \mathbb{T}^2(X/T) \longrightarrow & \cdots
\end{array}
$$

which leads to an exact sequence

$$0 \longrightarrow \mathbb{T}^1(Y/T) \longrightarrow \mathbb{T}^1(X/T) \longrightarrow H^0(\mathcal{T}^1_{X/T}) \longrightarrow \mathbb{T}^2(Y/T) \longrightarrow \mathbb{T}^2(X/T) \longrightarrow \cdots$$

If in addition $\mathcal{T}^2_{X/T} = 0$, then ... can be replaced by 0.

4. Liftings of small resolutions

By a *small resolution* of a scheme X we mean a smooth modification $\pi : Y \longrightarrow X$ with the property that the exceptional set has codimension ≥ 2 in Y. Only very special singularities do admit a small resolution and in this section we concentrate on the ordinary double point.

Let us study first the local situation. We put $X := Spec(R)$, where $R := k[[x, y, z, t]]/(f)$, where $f = xy - zt$. Let $Z \subset X$ be the closed subscheme defined by the ideal $(x, z) \subset R$ and let $Y := B_Z(X)$ be the blow-up of X in Z. Y is described as sub-scheme of $Spec(k[[x, y, z, t]]) \times \mathbb{P}^1$ by the equations

$$xy - zt = 0, \quad xu - zv = 0, \quad (u : v) \in \mathbb{P}^1$$

The exceptional locus is a copy of $\mathbb{P}^1$, with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. In [8] one finds the following basic calculation.

Theorem 4.1. (Friedmann) *Let* $\pi : Y \longrightarrow X$ *be a small resolution of the node in dimension three. Then*:

(1) $\mathcal{O}_X = \pi_*(\mathcal{O}_Y)$ *and* $R^i \pi_*(\mathcal{O}_Y) = 0$ *for* $i \geq 1$.
(2) $\Theta_{X/k} = \pi_*(\Theta_{Y/k})$ *and* $R^i \pi_*(\Theta_{Y/k}) = 0$ *for* $i \geq 1$.

From this, one deduces $H^1(\Theta_{Y/k}) = H^2(\Theta_{Y/k}) = 0$. This implies that Y lifts over any S and does so in a *unique* manner. Explicitly one can construct this lifting of Y over S as follows. We put

$$\mathcal{X} = Spec(\mathcal{R}), \quad \mathcal{R} := \mathcal{O}_S[[x, y, z, t]]/(xy - zt)$$

and let

$$\mathcal{Y} = B_Z(\mathcal{X}), \quad Z = V(x, z) \subset \mathcal{X}$$

Note that

$$\mathbb{T}^1(X/k) = H^0(X, \mathcal{T}_{X/k}^1) = k[[x, y, z, t]]/(J_f, f) = k[[x, y, z, t]]/(x, y, z, t) = k$$

so the lifting of X is not unique, rather there is a one-dimensional space of choices.

The lifting of X over which there also is a lifting of Y can be characterized by having a *singular section* $\sigma : S \longrightarrow \mathcal{X}$ induced by the canonical projection

$$\mathcal{R} \longrightarrow \mathcal{O}_S, \quad x, y, z, t \mapsto 0$$

This is to be contrasted with the lifting of X to

$$\mathcal{X} = Spec(\mathcal{R}), \quad \mathcal{R} = \mathcal{O}_S[[x, y, z, t]]/(xy - zt - q)$$

where $\mathcal{O}_S/(q) = k$. As a result of this local analysis one finds the following

Proposition 4.2. *Let X be a scheme having $\Sigma \subset X$ as set of nodes and let $\pi : Y \longrightarrow X$ be a small resolution. Let $\mathcal{Y}$ be any lift of Y over S and let $\mathcal{X}$ be the blow down of $\mathcal{Y}$. Then for each $x \in \Sigma$ there is a singular section $\sigma : S \longrightarrow \mathcal{X}$ passing through x.*

As a corollary we obtain the following theorem that forms the basis for our construction of non-liftable Calabi-Yau spaces. Recall that X/k is called *rigid* if $\mathbb{T}^1(X/k) = 0$.

Theorem 4.3. *Let $\mathcal{X}$ be a flat and proper scheme over $S = Spec(A)$ of relative dimension 3, A a complete noetherian domain with residue field k and fraction field $K = Q(A)$. Assume that*:

(1) *The generic fibre $X_\eta := \mathcal{X} \otimes_A K$ is smooth.*
(2) *The special fibre $X := \mathcal{X} \otimes_A k$ is rigid with nodes as singularities.*

Let $\pi : Y \longrightarrow X$ be a small resolution. Then Y does not lift to S.

Proof. Let $A_n := A/\mathbf{m}^{n+1}$ and put $S_n := Spec(A_n)$, $X_n := \mathcal{X} \times_S S_n \longrightarrow S_n$, so that $X = X_0$. Assume that Y has a lift $\mathcal{Y}$ over S. From it we get a lift $Y_n := \mathcal{Y} \times_S S_n$ of Y. Then the blow-down of $Y_n \longrightarrow S_n$ is a lift of X, which by the proposition 4.2 comes with singular sections through the singularities. As X is rigid, the lifting is unique, hence must be isomorphic to the given lifting $X_n \longrightarrow S_n$. Hence, for any n the morphism $X_n \longrightarrow S_n$ has a singular section. The union of singular sections (corresponding to the nodes of X) is the zero set of the Fitting ideal $\mathrm{Fitt}_3\, \Omega^1_{X_n/S_n}$. As the formation of the Fitting ideal is compatible with base change, the Fitting ideal $\mathrm{Fitt}_3\, \Omega^1_{\mathcal{X}/S}$ is non-trivial. This contradicts the smoothness of the generic fibre of $\mathcal{X} \longrightarrow S$. Hence Y can not lift over S.   □

Corollary 4.4. *Under the above assumptions, if $A = \mathbb{Z}_p$, then Y has no lift to characteristic zero.*

Proof. Let $\mathcal{Y} \longrightarrow S$ is a lift of Y, where S is a complete DVR which is a finite extension of $\mathbb{Z}_p$. We get by blowing down a family $\mathcal{X}' \longrightarrow S$ and a map $\mathcal{Y} \longrightarrow \mathcal{X}'$ over S. Pulling back the given family $\mathcal{X} \longrightarrow Spec(\mathbb{Z}_p)$ via the finite map $Spec(S) \longrightarrow Spec(\mathbb{Z}_p)$ we get a family $\mathcal{X}'' \longrightarrow S$. By rigidity of X the lifting is unique and we conclude that $\mathcal{X}'$ and $\mathcal{X}''$ are isomorphic over S. We now get a contradiction to the Theorem 4.3.   □

Remark 4.5. The theorem can be understood in terms of $\mathbb{T}^i$ as follows. As X is assumed to have only nodes as singularities, one has $\mathcal{T}^2_{X/k} = 0$ and we have an exact sequence

$$0 \longrightarrow \mathbb{T}^1(Y/k) \longrightarrow \mathbb{T}^1(X/k) \longrightarrow H^0(X, \mathcal{T}^1_{X/k}) \longrightarrow \mathbb{T}^2(Y/k) \longrightarrow \mathbb{T}^2(X/k) \longrightarrow 0$$

If furthermore X/k is rigid, we see that Y/k is rigid as well. The sheaf $\mathcal{T}^1_{X/k}$ is supported on the singularities and each node contributes a copy of k to $H^0(X, \mathcal{T}^1_{X/k})$. Any lifting of X that is "smoothing out" a node will produce a non-zero element of $H^0(X, \mathcal{T}^1_{X/k})$, which in turn gives a non-zero element in $\mathbb{T}^2(Y/k)$, namely the obstruction to lift Y over that lifting of X. That obstruction element maps to zero in $\mathbb{T}^2(X/k)$, as it should.

5. Projective examples

We give some examples where the ideas of the previous sections can be applied.

5.1. A projective Calabi-Yau threefold over $\mathbb{F}_3$ with no lifting

In this section we shall give an example of a Calabi-Yau manifold over $\mathbb{Z}/3$ which has no lifting to characteristic zero.

Proposition 5.1. *There exists a projective scheme $\mathcal{X}$ flat over $\mathbb{Z}$ such that*

(1) *the generic fiber $X := \mathcal{X}_\eta$ is a rigid Calabi-Yau manifold with $h^{1,1}(X) = 40$, $h^{12}(X) = 0$,*
(2) *the reduction X_3 of $\mathcal{X}$ modulo 3 has two nodes as the only singularities.*
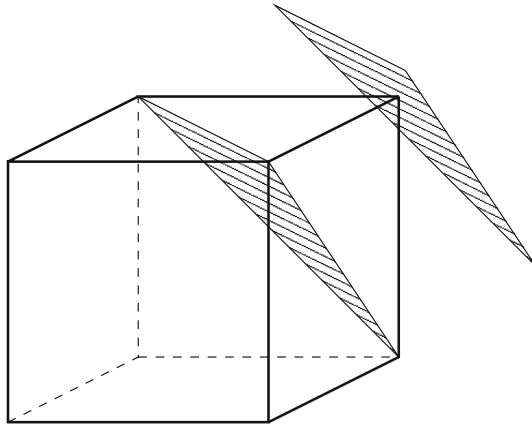
Fig. 1. Octic arrangement with bad reduction at 3

We shall now describe the construction of $\mathcal{X}$. Let $D \subset \mathbb{P}^3$ be the arrangement of eight planes given by the following equation

$$(x - t)(x + t)(y - t)(y + t)(z - t)(z + t)(x + y + z - t)(x + y + z - 3t) = 0.$$

This is the arrangement no 86^a from the paper [5], it consist of six planes of a cube and two additional planes: one contains three vertices of the cube but no edges, the other contains exactly one vertex of the cube and is parallel to the first one. In fact there are two choices for the second plane and we make a specific choice, as suggested in the picture (Fig. 1).

Let $\mathcal{X}$ be a scheme over $\mathbb{Z}$ given by a resolution of singularities of the double cover of $\mathbb{P}^3$ branched along this octic surface; we choose a resolution which is induced by blowing-up fourfold points and then double lines of the octic surface in $\mathbb{P}^3$ as described in [5]. The generic fibre $X := \mathcal{X}_\eta$ is a rigid Calabi-Yau manifold with topological Euler characteristic $e(X) = 80$ and Hodge numbers

$$h^{11} = 40, \quad h^{12} = 0.$$

The reduction X_3 of $\mathcal{X}$ modulo 3 turns out to be singular. To find the singularities of X_3 we have to compare the singularities of the arrangement D and its reduction D_3 modulo 3. The only singularities of D are 28 double lines, they intersects in 10 fourfold points (of type p_4^0 in the notations introduced in [5]), the reduction D_3 has *one additional fourfold point*, as in characteristic 3 the plane $x + y + z - 3t = 0$ contains the two vertices $(1, 1, 1, 1)$ and $(-1, -1, -1, 1)$ of the cube.

To study $\mathcal{X}$ and its reduction X_3, we have to study the resolution process of the double cover. Our strategy is to blow up in $\mathbb{P}^3$ to make the strict transform of the divisor D non-singular. This can be done by first blowing-up the 10 fourfold points and then sequentially the double lines. In characteristic 3 the surface D has an extra fourfold point, but we still blow-up first the ten fourfold points and then the double lines. At the eleventh fourfold point, after blowing-up the first double line through it, the strict transforms of the two planes that do not contain that line

intersect along a *cross* (two intersecting lines). When we blow-up the cross we end up with a threefold with one node and a sum of eight smooth, pair-wise disjoint surfaces which do not contain the node.

In local coordinates we can assume that the four planes we are considering have equations $x = 0$, $y = 0$, $z = 0$, $x + y + z - 3 = 0$. In characteristic zero we blow-up the double lines. Let us first blow-up the two disjoint lines $x = y = 0$ and $z = x + y + z - 3 = 0$. In one of the affine charts the blow-up of $\mathbb{P}^3$ is given by the equation $(v - 1)z = x(y + 1) - 3$. The threefold is smooth but in characteristic 3 it acquires a node at $x = 0$, $y = -1$, $z = 0$, $v = 1$. Since the surface $x = 0$, $z = 0$ is a Weil divisor on the threefold which is not Cartier (it is a component of the exceptional locus of the blow–up) the node admits a projective small resolution.

The double cover of the threefold branched along the sum of surfaces, is the reduction X_3 of $\mathcal{X}$ modulo 3. It has two nodes, which form the preimage of the double point under the double cover. There exists a projective small resolution X_3 of the nodes which is a smooth rigid Calabi-Yau manifold over $\mathbb{Z}/3$. The projective small resolution can be obtained directly from the reduction D_3 of the arrangement D modulo 3 by blowing-up first all eleven fourfold points and then the double lines.

Corollary 5.2. *There exists a Calabi-Yau manifold Y_3 defined over $\mathbb{Z}/3$ which has no lifting to characteristic zero. It has the following Hodge numbers $h^{11} = 42$, $h^{12} = 0$.*

Using the formulas from [5,6] we can compute the Hodge numbers $h^{11} = 42$, $h^{12} = 0$, so indeed Y_3 is also rigid. As the nodal X_3 is the reduction of $\mathcal{X}$ with smooth general fibre $\mathcal{X}_\eta$, we can apply the Theorem 4.3 to conclude that Y_3 can not be lifted to characteristic zero.

Remark 5.3. A local analysis learns that the nodes of X_3 are lifted to

$$(\mathbb{Z}/9)[[x, y, z, t]]/(xy - zt - 3),$$

which shows that Y_3 does not lift to any ring in which $3 \neq 0$.

5.2. A Calabi-Yau threefold over $\mathbb{F}_5$ with an obstructed deformation

In this section we shall prove the following proposition

Proposition 5.4. *There exists a projective scheme $\mathcal{X}$ flat over $\mathbb{Z}[\frac{-1+\sqrt{5}}{2}]$ such that*

(1) *the generic fiber $X := \mathcal{X}_\eta$ is a rigid Calabi-Yau manifold with $h^{1,1}(X) = 38$, $h^{12}(X) = 0$,*

(2) *the reduction X_5 of $\mathcal{X}$ modulo the unique prime ideal in $\mathbb{Z}[\frac{-1+\sqrt{5}}{2}]$ containing (5) is a non–rigid Calabi-Yau manifold over $\mathbb{Z}/5$ that has an obstructed first order deformation $X_{5,\epsilon}$. Its Hodge numbers are $h^{11} = 39$, $h^{12} = 1$.*

Consider the octic arrangement $D \subset \mathbb{P}^3$ of eight planes given by (see Fig. 2)

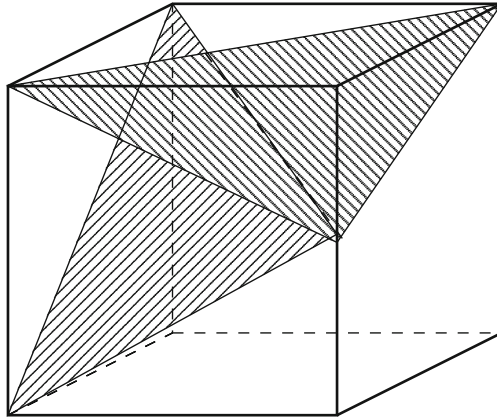$$(x-t)(x+t)(y-t)(y+t)(z-t)(z+t)(x + y + Az - At)(x - By - Bz + t) = 0.$$

Fig. 2. Octic arrangement with bad reduction at 5

For general A and B the arrangement has seven fourfold points (of type p_4^0), namely the three infinite points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ and four of the vertices of the cube, namely $(1, -1, 1, 1)$, $(-1, -1, 1, 1)$ contained in the plane $(x + y + Az - At) = 0$ and $(-1, 1, 1, 1)$, $(-1, 1, -1, 1)$ contained in the plane $(x - By - Bz + t) = 0$.

If there is a special relation between the parameters A and B, the line of intersection of these two planes intersect an edge of the cube, giving rise to an arrangement with *eight* fourfold point. For example, the two planes meet on the edge $y + t = z + t = 0$ precisely when $A + B = -1$ and on the edge $x - t = y - t = 0$ precisely when $AB = A + B$. In that case the resolution of the double cover branched along D defines a smooth Calabi-Yau manifold with $e = 72$

$$h^{11} = 37, \quad h^{12} = 1$$

as can be computed using [6]. Indeed, moving this extra fourfold point along the edge of the cube produces an equisingular deformation of the arrangement and leads to a one-parameter family of such Calabi-Yau threefolds.

If *both* conditions

$$A + B = -1, \quad AB = A + B$$

are satisfied, then A and B are roots of the equation

$$x^2 + x - 1 = 0$$

and the arrangement has *nine* fourfold points. Hence such an arrangement with nine fourfold points defines a projective scheme $\mathcal{X}$ flat over

$$\mathbb{Z}[x]/(x^2 + x - 1) \cong \mathbb{Z}\left[\tfrac{-1+\sqrt{5}}{2}\right]$$

The generic fibre $\mathcal{X}_\eta$ is a projective Calabi-Yau threefold with

$$h^{11} = 38, \quad h^{12} = 0.$$

Consider the reduction D_5 of D modulo 5. Using [6] one computes that one obtains again a smooth Calabi-Yau manifold X_5, now with

$$h^{11}(X_5) = 39, \quad h^{12}(X_5) = 1$$

The remarkable thing that happens is

$$x^2 + x - 1 = (x - 2)^2 \mod 5$$

so that

$$\mathbb{Z}[x]/(x^2 + x - 1) \otimes \mathbb{Z}/5 = \mathbb{Z}/5[\epsilon], \quad \epsilon^2 = 0$$

According to [6] the infinitesimal deformations $X_{5,\epsilon}$ of X_5 are given by equisingular infinitesimal deformations of D_5. Although the only solution to $x^2 + x - 1 = 0$ in $\mathbb{Z}/5$ is 2, in the ring of dual number $(\mathbb{Z}/5)[\epsilon]$, $(\epsilon^2 = 0)$, it has also the solution $2 + \epsilon$, which means that the infinitesimal equisingular deformations of D_5 (and hence also infinitesimal deformations of X_5) are given by arrangement D_ϵ given by the following equation

$$(x - t)(x + t)(y - t)(y + t)(z - t)(z + t)$$
$$\cdot (x + y + (2 + \epsilon)z - (2 + \epsilon)t)(x + (3 + \epsilon)y + (3 + \epsilon)z + t) = 0.$$

This first order deformation cannot be lifted to a second order deformation (cf. [24]) since 2 is the only solution of $x^2 + x - 1 = 0$ in the field $\mathbb{Z}/5$, whereas in the ring $(\mathbb{Z}/5)[\epsilon]/\epsilon^3$ its solutions are $2 + c \cdot \epsilon^2$ (and similar for higher orders).

Remark 5.5. Similarly one can show that X_5 has no lifting to $\mathbb{Z}/25$.

6. Non-projective examples

It is rather easy to produce examples of rigid Calabi-Yau spaces $\mathcal{X}$ over $\mathbb{Z}$ whose reduction $X = \mathcal{X}$ mod p aquires some extra nodes. However, in many cases there does not exist a small resolution $\pi : Y \longrightarrow X$ in the category of schemes. However, in the larger category of *algebraic spaces* Artin [1] has shown that small resolutions do exist. The arguments in the derivation of Theorem 4.3 were of very general nature and carry over verbatim to the more general context of algebraic spaces.

6.1. A Calabi-Yau space over $\mathbb{F}_5$ having no lift

We shall give an example of a Calabi-Yau space defined over $\mathbb{Z}/5$ that cannot be lifted to characteristic zero.

Proposition 6.1. *There exists a projective scheme $\mathcal{X}$ flat over $\mathbb{Z}$ such that*

(1) *the generic fiber $X := \mathcal{X}_\eta$ is a rigid Calabi-Yau manifold with $h^{1,1}(X) = 52, h^{12}(X) = 0$, its L series is equal to a twist of the unique level 5 cusp form.*
(2) *the reduction X_5 of $\mathcal{X}$ modulo 5 has a node as the only singularity*
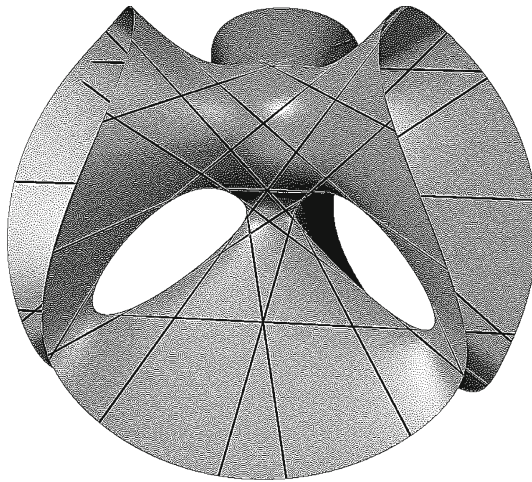
Fig. 3. Clebsch diagonal cubic

Consider the arrangement $D \subset \mathbb{P}^3$ given by the following equation

$$\left(x^3+y^3+z^3+t^3-(x+y+z+t)^3\right)(x+y)(y+z)(z+t)(x+y+z)(y+z+t) = 0.$$

It consists of the Clebsch diagonal cubic (see Fig. 3)

$$x^3 + y^3 + z^3 + t^3 - (x+y+z+t)^3 = 0$$

and five planes tangent at points where three lines lying on the cubic meet, these points are called *Eckardt points*. A smooth cubic in $\mathbb{P}^3$ is a del Pezzo surface, it has at most 10 Eckardt points, the diagonal Clebsch cubic is the only cubic in $\mathbb{P}^3$ that has exactly ten Eckardt points. One of them is the point $(1, -1, 0, 0)$, and the other can be obtained by permutation of coordinates (in $\mathbb{P}^4$). Every Eckardt point is contained in three lines lying on the surface, on any of these lines there is another Eckardt point. We choose five Eckardt points and five lines lying on the cubic such that the lines form a pentagon with vertices at the fixed Eckardt points.

The singularities of the surface D consists of ten fourfold points, ten double and five triple lines. The fourfold points fall into two groups

$$
\begin{aligned}
A_1 &= (1, -1, 0, 0) & \quad & B_1 = (1, -1, 1, -1) \\
A_2 &= (0, 1, -1, 0) & \quad & B_2 = (0, 1, -1, 1) \\
A_3 &= (0, 0, 1, -1) & \quad & B_3 = (-1, 0, 1, -1 \\
A_4 &= (0, 0, 0, 1) & \quad & B_4 = (1, -1, 0, 1) \\
A_5 &= (1, 0, 0, 0) & \quad & B_5 = (-1, 1, -1, 0)
\end{aligned}
$$

The triple lines are

$$
\begin{aligned}
l_1 &: \quad x + y = z + t = 0 \\
l_2 &: \quad x + y = z = 0 \\
l_3 &: \quad y + z = x = 0 \\
l_4 &: \quad y + z = t = 0 \\
l_5 &: \quad z + t = y = 0.
\end{aligned}
$$

The double lines also fall into two groups

$$
\begin{array}{lll}
m_1: & x+y = \quad t \quad = 0 \\
m_2: & y+z = x+t = 0 \\
m_3: & z+t = \quad x \quad = 0 \\
m_4: & x+z = \quad y \quad = 0 \\
m_5: & y+t = \quad z \quad = 0
\end{array}
\quad\Bigg\|\quad
\begin{array}{lll}
n_1: & x+y = \quad y+z \quad = 0 \\
n_2: & x+y = y+z+t = 0 \\
n_3: & y+z = \quad z+t \quad = 0 \\
n_4: & z+t = x+y+z = 0 \\
n_5: & x+y+z = y+z+t = 0
\end{array}
$$

The following tables describe incidences of points and lines

line	contains points	point	lies on lines
l_1	A_1, A_3, B_2	A_1	l_1, l_2, m_1, n_4
l_2	A_1, A_4, B_1	A_2	l_3, l_4, m_2, n_5
l_3	A_2, A_4, B_2	A_3	l_1, l_5, m_3, n_2
l_4	A_2, A_5, B_5	A_4	l_2, l_3, m_4, n_1
l_5	$A_3, A_5 B_3$	A_5	l_4, l_5, m_5, n_3
m_1	A_1, B_5	B_1	l_1, m_2, n_1, n_5
m_2	A_2, B_1	B_2	l_2, m_3, n_3, n_4
m_3	A_3, B_2	B_3	l_3, m_4, n_4, n_5
m_4	A_4, B_3	B_4	l_4, m_5, n_2, n_5
m_5	A_5, B_4	B_5	l_5, m_1, n_1, n_2
n_1	A_4, B_1, B_5		
n_2	A_3, B_4, B_5		
n_3	A_5, B_1, B_2		
n_4	A_1, B_2, B_2		
n_5	A_2, B_3, B_4		

Although D is not an octic arrangement in the sense of [5], the singularities of the double covering of $\mathbb{P}^3$ branched along D can be resolved in a similar manner. First we blow–up the fourfold points A_i and B_i. The lines intersecting at points B_i's are now disjoint. Since at the points A_i's one of the planes was threefold tangent to the cubic, after blowing up that point the strict transform of the cubic and the tangent plane are still tangent along a line. This line intersects the other two planes that contained the fourfold point. We have to blow-up the line, then the strict transforms are transversal so we have to blow-up a line again.

After this we end up with 10 double and 5 triple lines, these lines are pairwise disjoint. We blow-up them all, since we add the exceptional divisors corresponding to the triple lines to the branch locus, we get 15 additional double lines which we have to blow-up again.

The above resolution of singularities of the double covering of $\mathbb{P}^3$ branched along D defines a projective scheme $\mathcal{X}$ over $\mathbb{Z}$ whose generic fibre $X := \mathcal{X}_\eta$ is a Calabi-Yau manifold. Now, we shall compute the Euler characteristic of X, since we blow-up $\mathbb{P}^3$ ten times at a point and at 40 times at a line—for every point A_i

we blow-up twice at the intersection of the strict transform of cubic at tangent plane, there are 10 double and 5 triple lines, but every triple lines is resolved by blowing–up four times at a line. Consequently $e(\tilde{\mathbb{P}^3}) = 4 + 10 \cdot 2 + 40 \cdot 2 = 104$.

The final branch locus is a blow-up of a sum of a cubic, five planes and five products $\mathbb{P}^1 \times \mathbb{P}^1$. During blow-up of every fourfold point we blow–up a cubic and three planes, moreover blowing-up every line of intersection of a strict transform of the cubic and tangent plane we blow-up two planes. So we get $e(D^*) = 9 + 5 \cdot 3 + 5 \cdot 2 \cdot 2 + 10 \cdot 4 + 10 \cdot 2 = 104$.

The Calabi-Yau manifold X is rigid, since the resolution of the singularities was obtained by blowing-up points and lines, using [6], it is enough to show, that every equisingular deformation of D is trivial. Observe that the points B_i and the lines L_i, m_i, n_i are all determined by the points A_i. Indeed intersection of the plane containing three consecutive vertices of the pentagon formed by the points A_i with the line containing the other two vertices gives one of the points B_i, so we can determine all the fourfold points, then also the lines are determined. Since all the choices of five points in general position in $\mathbb{P}^3$ are projectively equivalent, it is enough to check with a computer Singulae ([]) that there exists exactly one cubic surface which contains all the lines.

So we have $e(X) = 104, h^{11} = 52, h^{12} = 0$. Since during the resolution we blow-up $\mathbb{P}^3$ fifty times, the rank of the subgroup of the Picard group that is invariant w.r.t the involution is 51. We can use the standard method of counting points modulo small primes to show that the Calabi-Yau manifold X is modular, and the corresponding modular form has level 720. If we multiply the equation of the branch locus by 3, then we get a rigid Calabi-Yau manifold with the same Hodge numbers and L-series of level five, exactly the same as the Schoen self product of semistable elliptic fibrations corresponding to $\Gamma_0^0(5)$. The relation to prime 5 is not surprising, as the Clebsch cubic is known to be a model of the Hilbert modular surface for $\mathbb{Q}(\sqrt{5})$ [15].

Corollary 6.2. *There exist a Calabi-Yau space Y_5 defined over $\mathbb{Z}/5$ that has no lifting to characteristic zero. Moreover we have $h^{11}(Y_5) = 53, h^{12}(Y_5) = 0$.*

The reduction of Clebsch cubic modulo 5 has a node at the point $(1, 1, 1, 1)$, so X_5, the reduction mod 5 of $\mathcal{X}$, gets a node that is not present in characteristic zero. Let Y_5 be a small resolution of the node on X_5. We compute the Hodge numbers using formulas in [6], non-liftability follows from Theorm 4.3.

The Calabi-Yau manifold Y_5 is not projective. The projectivity of Y_5 would imply existence of a prime Cartier divisor D on Y_5 which has positive intersection $(DL) > 0$ with the exceptional line L. On Y_5 we have a K3 fibration given by resolutions of double coverings of planes from the pencil $\lambda(x + y) = \mu(z + t)$ branched along the intersection with the sextic surface $(x^3 + y^3 + z^3 + t^3 - (x + y + z + t)^3)(y + z)(x + y + z)(y + z + t) = 0$. The Picard group of Y_5 is generated by the Picard group of the (scheme theoretic) generic fiber of K3 fibration and components of reducible fibers. Direct computations show that the Picard group of Y_5 is equal to the pullback of the Picard group

of X_5 hence it is orthogonal to the line L, which contradicts existence of D with $(DL) > 0$.

There is another arrangement with the Clebsch cubic that leads to a non-liftable (but non-projective) Calabi-Yau space in characteristic 5.

Consider the octic given in $\mathbb{P}^4$ by the following equations

$$\left(x^3 + y^3 + z^3 + t^3 + u^3\right) xyztu = 0$$

$$x + y + z + t + u = 0$$

The cubic intersects planes in 15 lines with equations $x + y = z + t = u = 0$ and then permuting. There are another double lines coming from intersection of planes $x = y = z + t + u = 0$. There are ten fourfold points which are intersection of three lines and the cubic, they have coordinates $(1, -1, 0, 0, 0)$ and permuted. The resolution comes from blowing-up first fourfold points and then 25 double lines. The Euler characteristic of the smooth model is 84, the Hodge numbers $h^{1,1} = 42$, $h^{1,2} = 0$. We blow-up 35 times so there are 6 skew-symmetric divisors, they come from contact planes with equations $x + y = z + t + u = 0$ and symmetric. In characteristic five there is additional node but again we are not able to find a projective small resolution.

6.2. Non-liftable Calabi-Yau spaces coming from fiber products

In this section we shall consider Calabi-Yau spaces that are constructed as desingularization of fiber products of semi-stable, rational elliptic surfaces with section. This construction originally goes back to Schoen [19]. More precisely we consider twisted self-fiber products of semistable elliptic surfaces with four singular fibers [23].

Proposition 6.3. *For* $p = $ *3,5,7,11,17,29,41,73, 251, 919, 9001 there exists an algebraic space over a finite ring extension R of $\mathbb{Z}$ such that*

(1) *the generic fiber $X := \mathcal{X}_\eta$ is a rigid Calabi-Yau manifold, it can be defined as a small resolution of a fiber product of two Beauville surfaces*
(2) *the reduction X_p of $\mathcal{X}$ modulo a prime ideal $\mathfrak{p}$ in R containing (p) has a node as the only singularity.*

There are six types of semi-stable rational elliptic surfaces with four singular fibres, the well-known Beauville surfaces [3]. Let S_1 and S_2 be two such surfaces. Assume that the singular fibers corresponds to points $0, 1, \lambda, \infty$ and $0, 1, \mu, \infty$, assume that the fibers $(S_1)_\lambda$ and $(S_2)_\mu$ have type I_1. Then the fiber product has a (non-projective) small resolution which is a rigid Calabi-Yau manifold. Assume that p is a prime number such that λ and μ have the same reduction mod p (different from 0 and 1). Then the reduction X_p of X mod p has an additional node and hence its small resolution Y_p is a Calabi-Yau space in characteristic p that has no lifting to characteristic zero.

We list some of the surfaces (the singular fibers and their positions) that we shall use in our constructions

Twist of $S(\Gamma_1(5))$			
0	1	∞	$\frac{1}{2}(55\sqrt{5}-123)$
I_5	I_1	I_5	I_1
Twists of $S(\Gamma_1(6))$			
0	1	∞	$\frac{8}{9}$
I_3	I_6	I_2	I_1
0	1	∞	9
I_2	I_3	I_6	I_1
0	1	∞	-8
I_3	I_2	I_6	I_1
Twists of $S(\Gamma_0(8) \cap \Gamma_1(4))$			
0	1	∞	1
I_2	I_1	I_8	I_1
0	1	∞	2
I_1	I_2	I_8	I_1
0	1	∞	$\frac{1}{2}$
I_8	I_2	I_1	I_1

Corollary 6.4. *For* $p = 3,5,7,11,17,29,41,73, 251, 919, 9001$ *there exists a Calabi-Yau space over* $\mathbb{Z}/p$ *that cannot be lifted to characteristic zero.*

A small resolution of the fiber product of first two surfaces is a rigid Calabi-Yau manifold X with Euler number $e(X) = 2(5 \times 3 + 5 \times 6 + 1 \times 6) = 102$ and the Hodge numbers $h^{11} = 51$, $h^{12} = 0$.
    Since

$$\frac{1}{2}\left(55\sqrt{5} - 123\right) - \frac{8}{9} = \frac{55 \times 9\sqrt{5} - (16 + 123 \times 9)}{18}$$

$$= \frac{-4 \times 9001}{18(55 \times 9\sqrt{5} + (16 + 123 \times 9))}$$

the reduction X_{9001} of X modulo 9001 is a rigid Calabi-Yau variety with one node. Its small resolution Y_{9001} is a smooth Calabi-Yau space over $\mathbb{Z}/9001$ that admits no lifting to characteristic 0.

Similarly

$$\frac{1}{2}\left(55\sqrt{5}-123\right)=9 \quad \text{in } \mathbb{Z}/29 \quad \text{and} \quad \mathbb{Z}/41$$

$$\frac{1}{2}\left(55\sqrt{5}-123\right)=-8 \quad \text{in } \mathbb{Z}/919$$

$$\frac{1}{2}\left(55\sqrt{5}-123\right)=-1 \quad \text{in } \mathbb{Z}/11$$

$$\frac{1}{2}\left(55\sqrt{5}-123\right)=2 \quad \text{in } \mathbb{Z}/251$$

$$\frac{8}{9}=9 \quad \text{in } \mathbb{Z}/73$$

$$\frac{8}{9}=-8 \quad \text{in } \mathbb{Z}/5$$

$$\frac{8}{9}=-1 \quad \text{in } \mathbb{Z}/17$$

$$\frac{8}{9}=\frac{1}{2} \quad \text{in } \mathbb{Z}/7$$

$$-1=2 \quad \text{in } \mathbb{Z}/3$$

Remark 6.5. When we uploaded the first version of our paper to the arxiv preprint server, we were informed that the same examples of fiber product Calabi-Yau manifolds were studied by Schoen [20]. He proved that they have third Betti number equal zero which implies non-liftability.

References

[1] Artin, M.: Algebraic construction of Brieskorn's resolutions. J. Algebra **29**, 330–348 (1974)

[2] Atiyah, M.: On analytic surfaces with double points. Proc. R. Soc. Lond. Ser. A **247**, 237–244 (1958)

[3] Beauville, A.: Les familles stables de courbes elliptiques sur P^1 admettant quatre fibres singulières. (French) [The stable families of elliptic curves on P^1 with four singular fibers]. C. R. Acad. Sci. Paris Sér. I Math. **294**(19), 657–660 (1982)

[4] Brieskorn, E.: Über die Auflösung gewisser Singularitäten von holomorphen Abbildungen. Math. Ann. **166**, 76–102 (1966)

[5] Cynk, S., Meyer, C.: Geometry and arithmetic of certain Double octic Calabi-Yau manifolds. Can. Math. Bull. **48**(2), 180–194 (2005)

[6] Cynk, S., van Straten, D.: Infinitesimal deformations of double covers of smooth algebraic varieties. Math. Nachr. **279**(7), 716–726 (2006)

[7] Ekedahl, T.: On non-liftable Calabi-Yau threefolds (preprint). math.AG/0306435

[8] Friedman, R.: Simultaneous resolution of threefold double points. Math. Ann. **274**(4), 671–689 (1986)

[9] van der Geer, G.: On the height of Calabi-Yau varieties in positive characteristic. Doc. Math. **8**, 97–113 (2003)

[10] Grothendieck, A.: Géométrie formelle et géometrie algébrique. Sem. Bourbaki, Exposé **182** (1959)

[11] Grothendieck, A.: Revetements Étales et Group Fondamental (SGA1). Séminaire de géométire algébrique du Bois Marie 1960–1961. Lecture Notes in Mathematics, vol. 224. Springer, Berlin (1971)

[12] Hirokado, M.: A non-liftable Calabi-Yau threefold in characteristic 3. Tohoku Math. J. (2) **51**(4), 479–487 (1999)

[13] Hirokado, M., Ito, H., Saito, N.: Calabi-Yau threefolds arising from fiber products of rational quasi-elliptic surfaces, I. Arkiv Mat. **45**(2), 279–296 (2007)

[14] Hirokado, M., Ito, H., Saito, N.: Calabi-Yau threefolds arising from fiber products of rational quasi-elliptic surfaces, II. Manuscripta Math. **125**(3), 325–343 (2008)

[15] Hirzebruch, F.: Hilberts modular group of the field $\mathbb{Q}(\sqrt{5})$ and the cubic diagonal surface of Clebsch and Klein. Russ. Math. Surv. **31**(5), 96–110 (1976)

[16] Illusie, L.: Complex Cotangent et déformations. Lecture Notes in Mathematics, vol. 239. Springer, Berlin (1971)

[17] Illusie, L.: Grothendieck's existence theorem in formal geometry (with a letter of J-P. Serre). In: Fantechi, B., Goettsche, L., Illusie, L., Kleiman, S., Nitsure, N., Vistoli, A. (eds.) Advanced School in Basic Algebraic Geometry, ICTP Trieste, 2003, Fundamental Algebraic Geometry, Grothendieck's FGA explained. Mathematical Surveys and Monographs, vol. 123. American Mathemcatical Society, Providence (2005)

[18] Kodaira, K.: On stability of compact submanifolds of complex manifolds. Am. J. Math. **85**, 79–94 (1963)

[19] Schoen, C.: On fiber products of rational elliptic surfaces with Section. Math. Z. **197**, 177–199 (1988)

[20] Schoen, C.: Desingularized fiber products of semi-stable elliptic surfaces with vanishing third Betti number (preprint). arXiv:0804.1078

[21] Schröer, S.: Some Calabi-Yau threefolds with obstructed deformations over the Witt vectors. Compos. Math. **140**(6), 1579–1592 (2004)

[22] Schröer, S.: The T^1-lifting theorem in positive characteristic. J. Algebraic Geom. **12**(4), 699–714 (2003)

[23] Schütt, M.: New examples of modular rigid Calabi-Yau threefolds. Collect. Math. **55**(2), 219–228 (2004)

[24] Wahl, J.: Simultaneous resolution and discriminantal loci. Duke Math. J. **46**(2), 341–375 (1979)